An Example of Double Cross Coproducts with Non-trivial Left Coaction and Right Coaction in Strictly Braided Tensor Categories *

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Abstract

An example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories is given.
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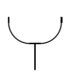
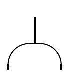
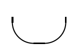
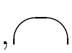
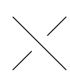
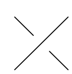
0 Introduction and Preliminaries

The double cross coproducts in braided tensor categories have been studied by Y.Bespalov, B.Drabant and author in [2] [12]. However, hitherto any examples of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories (i.e. the braiding is not symmetric) have not been found. Therefore Professor S.Majid asked if there is such example.

In this paper we first give the cofactorisation theorem of Hopf algebras in braided tensor categories. Using the cofactorisation theorem and Sweedler four dimensional Hopf algebra, we construct such example.

We denote the multiplication, comutiplication, evaluation d, coevaluation b, braiding and inverse braiding by

 ,  ,  ,  ,  and  ,

respectively. For convenience, we denote the inverse of morphism f by $\overline{f}$ if f has an inverse.

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Since every braided tensor category is always equivalent to a strict braided tensor category by [12, Theorem 0.1], we can view every braided tensor category as a strict braided tensor category and use braiding diagrams freely.

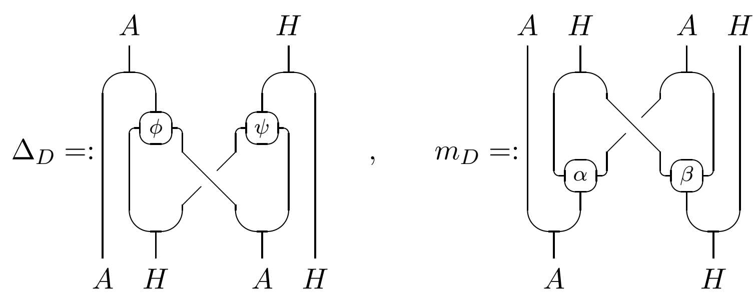
1 The cofactorisation theorem of bialgebras in braided tensor categories

Throughout this section, we work in braided tensor category $(\mathcal{C}, C)$ and assume that all Hopf algebras and bialgebras are living in $(\mathcal{C}, C)$ unless otherwise stated. We give the cofactorisation theorem of bialgebras in braided tensor categories in this section.

We first recall the double bicrossproducts in [12]. Let H and A be two bialgebras in braided tensor categories and

$$\alpha : H \otimes A \rightarrow \quad A \quad , \qquad \beta : H \otimes A \rightarrow H,$$
$$\phi : A \rightarrow H \otimes \quad A \quad , \qquad \psi : H \rightarrow H \otimes A$$

morphisms in $\mathcal{C}$.



and $\epsilon_D = \epsilon_A \otimes \epsilon_H$, $\eta_D = \eta_A \otimes \eta_H$. We denote $(A \otimes H, m_D, \eta_D, \Delta_D, \epsilon_D)$ by

$$A^{\phi}_{\alpha} \bowtie^{\psi}_{\beta} H,$$

which is called the double bicrossproduct of A and H.

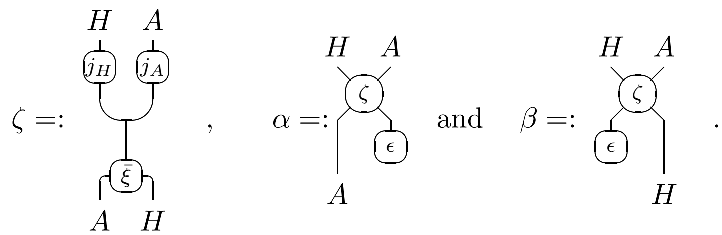
When ϕ and ψ are trivial, we denote $A^{\phi}_{\alpha} \bowtie^{\psi}_{\beta} H$ by $A_{\alpha} \bowtie_{\beta} H$. When α and β are trivial, we denote $A^{\phi}_{\alpha} \bowtie^{\psi}_{\beta} H$ by $A^{\phi} \bowtie^{\psi} H$. We call $A_{\alpha} \bowtie_{\beta} H$ a double cross product and denote it by $A \bowtie H$ in short. We call $A^{\phi} \bowtie^{\psi} H$ a double cross coproduct.

Theorem 1.1 *(Factorisation theorem) (See [9, Theorem 7.2.3]) Let X , A and H be bialgebras or Hopf algebras. Assume that j_A and j_H are bialgebra or Hopf algebra morphisms from A to X and H to X respectively. If $\xi =: m_X(j_A \otimes j_H)$ is an isomorphism from $A \otimes H$ onto X as objects in $\mathcal{C}$, then there exist morphisms*

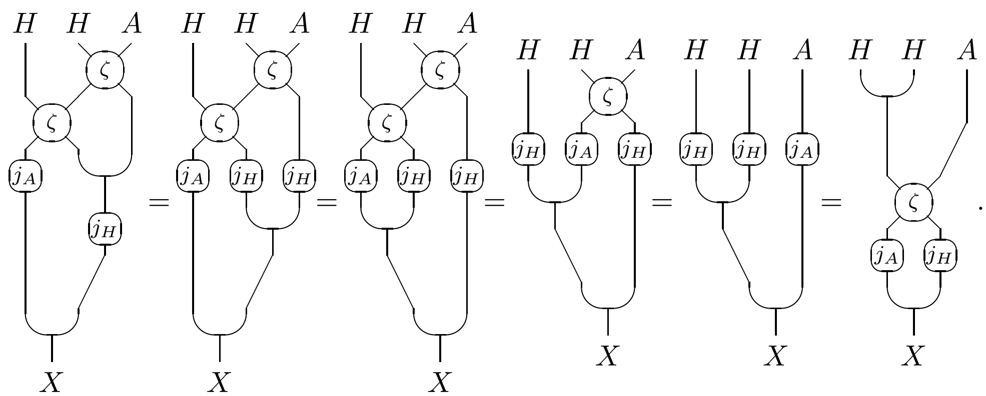
$$\alpha : H \otimes A \rightarrow A \quad \text{and} \quad \beta : H \otimes A \rightarrow H$$

such that $A_{\alpha} \bowtie_{\beta} H$ becomes a bialgebra or Hopf algebra and ξ is a bialgebra or Hopf algebra isomorphism from $A_{\alpha} \bowtie_{\beta} H$ onto X.

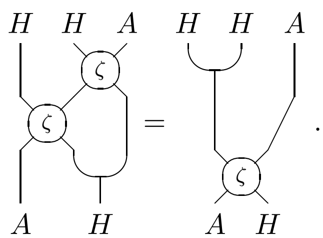
Proof. Set



We see

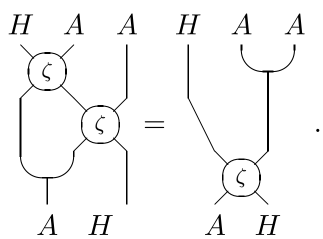


Thus



$\cdots\cdots(1)$

Similarly we have

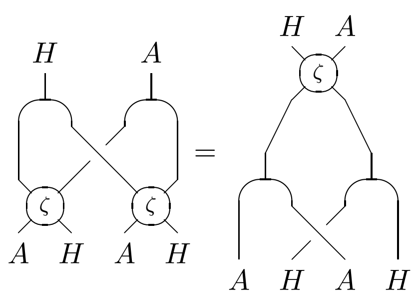


$\cdots\cdots(2)$

We also have

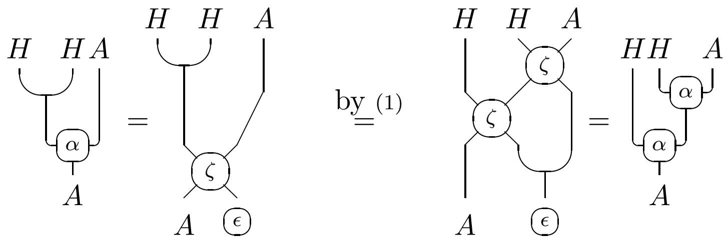
$$\zeta(\eta\otimes id)=id\otimes\eta \quad\text{and}\quad \zeta(id\otimes\eta)=\eta\otimes id\ . \qquad\cdots\cdots(3)$$

It is clear that ζ is a coalgebra morphism from $H\otimes A$ to $A\otimes H$, since j_A, j_H and m_X all are coalgebra homorphisms. Thus we have



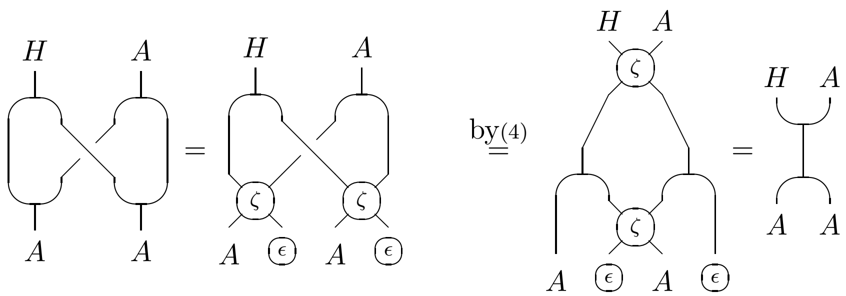
and $(\epsilon\otimes\epsilon)\zeta=(\epsilon\otimes\epsilon)$. $\qquad\cdots\cdots(4)$

We now show that (A,α) is an H-module coalgebra:



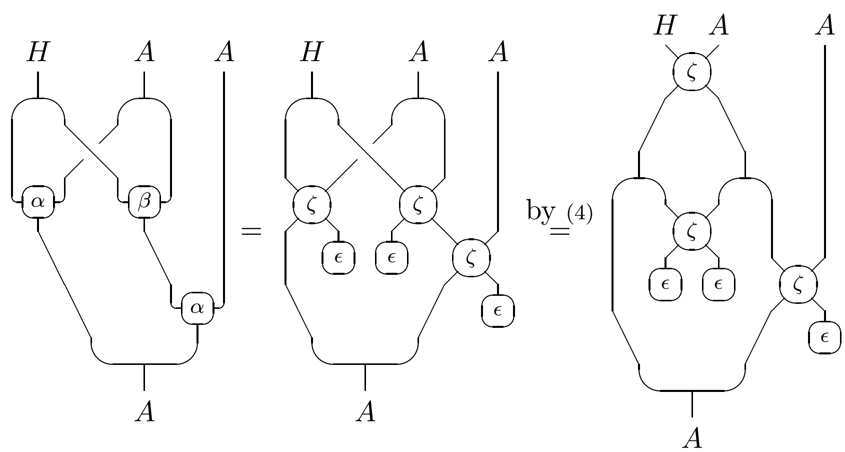
and $\quad \alpha(\eta\otimes id_A)=(id_A\otimes\epsilon)\zeta(\eta\otimes id_A)\overset{\text{by}(3)}{=} id_A.$

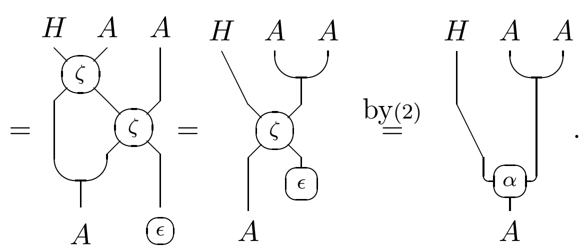
We see that $\epsilon\circ\alpha=(\epsilon\otimes\epsilon)\zeta\overset{\text{by}(4)}{=}\epsilon\otimes\epsilon \quad$ and



Thus (A,α) is an H-module coalgebra. Similarly, we can show that (H,β) is an A-module coalgebra.

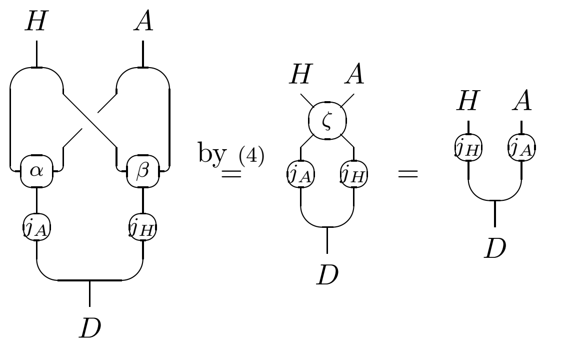
Now we show that conditions $(M1)$–$(M4)$ in [12,p37] hold. By (3), we easily know that$(M1)$ holds. Next we show that$(M2)$ holds.





Thus $(M2)$ holds. Similarly, we can get the proofs of $(M3)$ and$(M4)$. Consequently, $A_\alpha\bowtie_\beta H$ is a bialgebra or Hopf algebra by [12, Corollary 1.8]. It suffices to show that ζ is a bialgebra

morphism from$A_\alpha \bowtie_\beta H$ to X. Let $D = A_\alpha \bowtie_\beta H$. Since



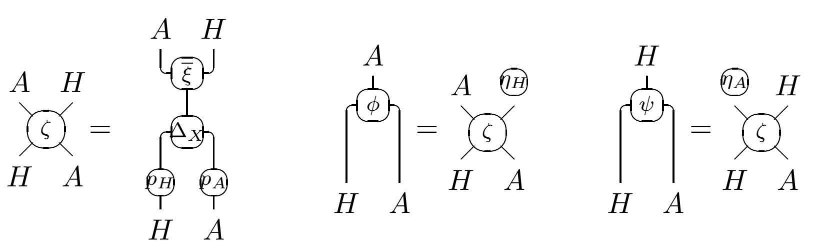
we have that ξ is a bialgebra morphism from $A_\alpha \bowtie_\beta H$ to X by [12, Lemma 2.5]. □

Theorem 1.2 *(Co-factorisation theorem) Let X , A and H be bialgebras or Hopf algebras. Assume that p_A and p_H are bialgebra or Hopf algebra morphisms from X to A and X to H, respectively. If $\xi = (p_A \otimes p_H)\Delta_X$ is an isomorphism from X onto $A \otimes H$ as objects in $\mathcal{C}$, then there exist morphisms:*

$$\phi : A \rightarrow H \otimes A \quad \text{and} \quad \psi : H \rightarrow H \otimes A$$

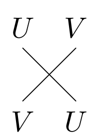
such that $A^\phi \bowtie^\psi H$ becomes a bialgebra or Hopf algebra and ξ is a bialgebra or Hopf algebra isomorphism from X to $A^\phi \bowtie^\psi H$.

Proof. Set



We can complete the proof by turning upside down the diagrams in the proof of the preceding theorem. □

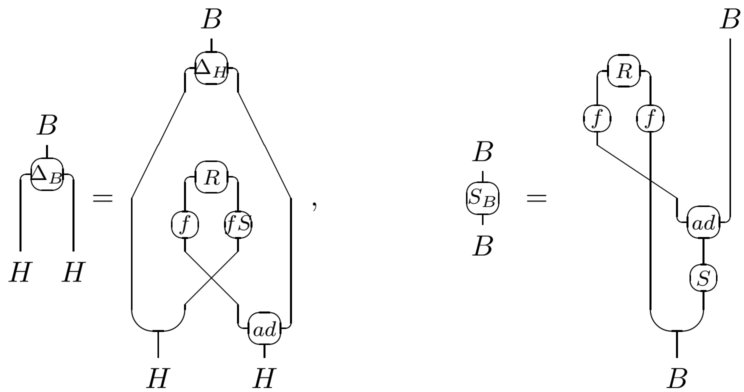
From now on, we always consider Hopf algebras over field k and the diagram



always denotes the ordinary twisted map: $x \otimes y \longrightarrow y \otimes x$. Our diagrams only denote homomorphisms between vector spaces, so two diagrams can have the additive operation.

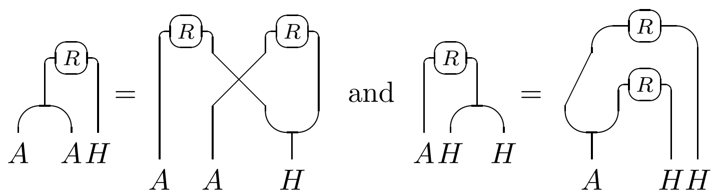
Let H be an ordinary bialgebra and (H_1, R) an ordinary quasitriangular Hopf algebra over field k. Let f be a bialgebra homomorphism from H_1 to H. Then there exists a bialgebra B, written as $B(H_1, f, H)$, living in $({}_{H_1}\mathcal{M}, C^R)$. Here $B(H_1, f, H) = H$ as algebra, its counit is ϵ_H,

and its comultiplication and antipode are



(see [8, Theorem 4.2]), respectively. In particular, when $H = H_1$ and $f = id_H$, $B(H_1, f, H)$ is a braided group, called the braided group analogue of H and written as $\underline{H}$.

R is called a weak R-matix of $A \otimes H$ if R is invertible under convolution with



Let (A, P) and (H, Q) be ordinary finite-dimensional quasitriangular Hopf algebras over field k. Let R be a weak R-matrix of $A \otimes H$. For any $U, V \in CW(A \otimes H) =: \{U \in A \otimes H \mid U$ is a weak R-matrix and in the center of $A \otimes H\}$,

$$R_D =: \sum R'P'U' \otimes Q'(R^{-1})''V'' \otimes P''(R^{-1})'V' \otimes R''Q''U''$$

is a quasitriangular structure of D and every quasitriangular structure of D is of this form ([3, Theorem 2.9]), where $R = \sum R' \otimes R''$, etc.

Lemma 1.3 *Under the above discussion, then*

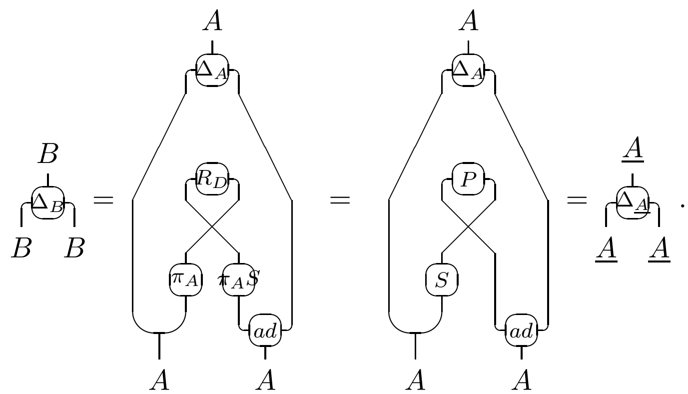
(i) $\pi_A : D \rightarrow A$ and $\pi_H : D \rightarrow H$ are bialgebra or Hopf algebra homomorphisms, respectively. Here π_A and π_H are trivial action, that is, $\pi_A(h \otimes a) = \epsilon(h)a$ for any $a \in A, h \in H$.

(ii) $B(D, \pi_A, A) = \underline{A}$ and $B(D, \pi_H, H) = \underline{H}$.

(iii) $\pi_{\underline{A}} : \underline{D} \rightarrow \underline{A}$ and $\pi_{\underline{H}} : \underline{D} \rightarrow \underline{H}$ are bialgebra or Hopf algebra homomorphisms, respectively.

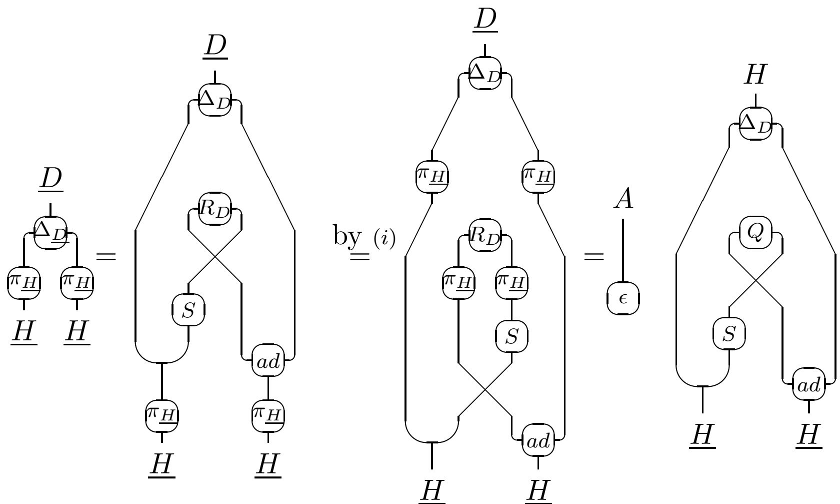
Proof. (i) It is clear.

(ii) It is enough to show $\Delta_B = \Delta_{\underline{A}}$ since $B = \underline{A}$ as algebras, where $B =: B(D, \pi_A, A)$. See



Thus $\Delta_B = \Delta_{\underline{A}}$. Similarly, we have $B(D, \pi_H, H) = \underline{H}$.

(iii) See

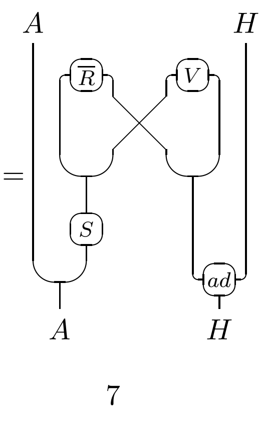


and $\epsilon \circ \pi_{\underline{H}} = \epsilon$. Thus $\pi_{\underline{H}}$ is a coalgebra homomorphism.

Since the multiplications in $\underline{D}$ and $\underline{H}$ are the same as in D and H, respectively, we have that $\pi_{\underline{H}}$ is an algebra homomorphism by (i). Similarly, we can show that $\pi_{\underline{A}}$ is a bialgebra homomorphism. □

We now investigate the relation among braided group analogues of quasitriangular Hopf algebras A and H and their double cross coproduct $D = A \bowtie^R H$.

Theorem 1.4 *Under the above discussion, let $\xi = (\pi_{\underline{A}} \otimes \pi_{\underline{H}})\Delta_{\underline{D}}$. Then*

$\xi =$ 

and ξ is surjective, where ad denotes the left adjoint action of H.

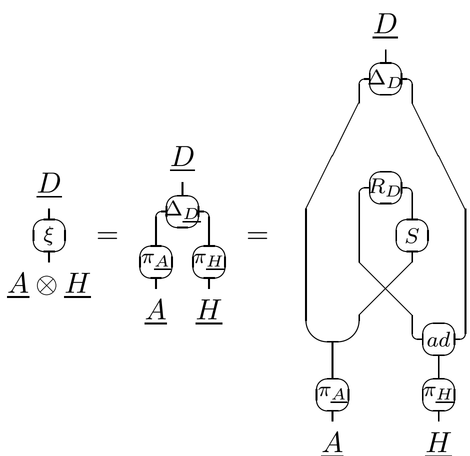
(ii) Furthermore, if A and H are finite-dimensional, then ξ is a bijective map from $\underline{D}$ onto $\underline{A} \otimes \underline{H}$. That is, in braided tensor category $({}_D\mathcal{M}, C^{R_D})$, there exist morphisms ϕ and ψ such that

$$\underline{D} \cong \underline{A}^{\phi} \bowtie^{\psi} \underline{H} \qquad (\text{ as Hopf algebras })$$

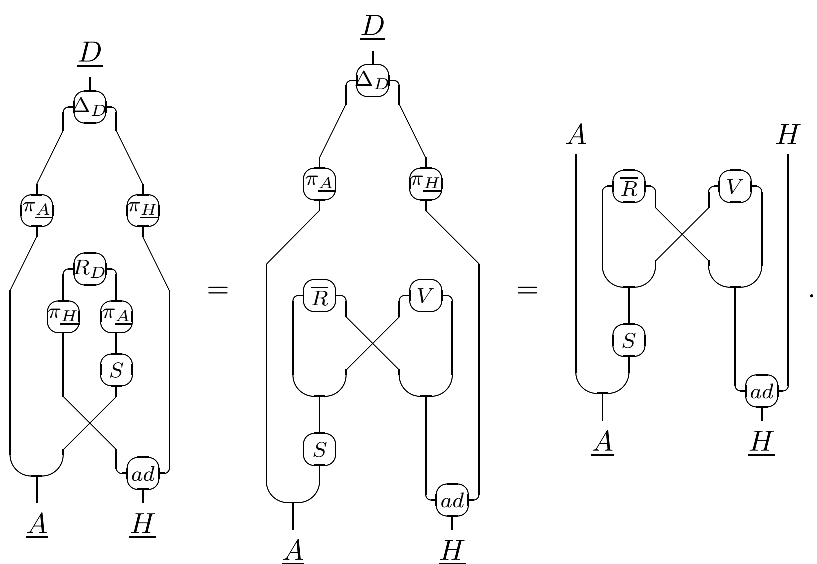
and the isomorphism is $(\pi_{\underline{A}} \otimes \pi_{\underline{H}})\Delta_{\underline{D}}$.

(iii) If H is commutative or $V = R$, then $\xi = id_{\underline{D}}$.

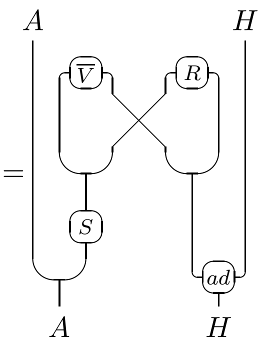
Proof. (i)



since π_A, π_H are algebra homomorphisms $=$



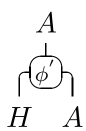
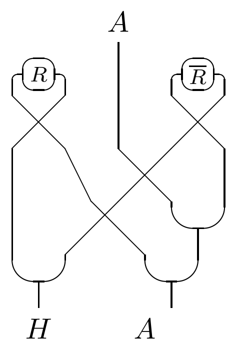
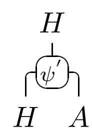
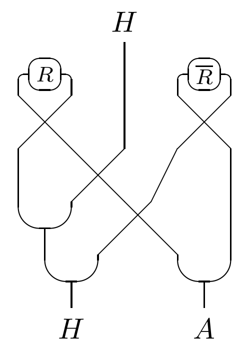
(ii) By the proof of (i), ξ is bijective and

$\bar{\xi} =$  .

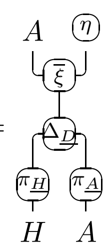
Applying the cofactorization theorem 1.2, we complete the proof of (ii).

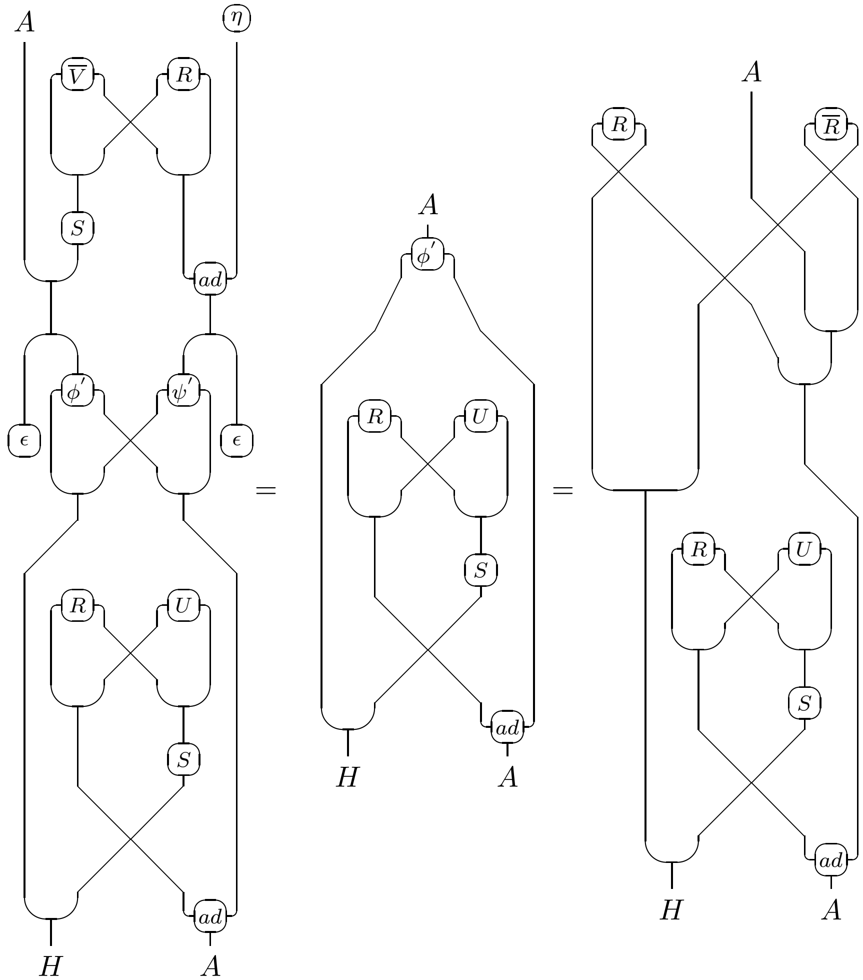
(iii) follows from (i). □

Remark. Under the assumption of Theorem 1.4, if we set

 $=:$  and  $=:$  ,

then $A^{\phi'} \bowtie^{\psi'} H = A \bowtie^{R} H$ by [3, Lemma 1.3]. We now see the relation among ϕ, ψ, R, ϕ' and ψ':

$\phi \overset{\text{by proof of Th 1.2}}{=} \zeta(id \otimes \eta) =$  $\overset{\text{by Th 1.4 (iii)}}{=}$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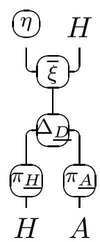
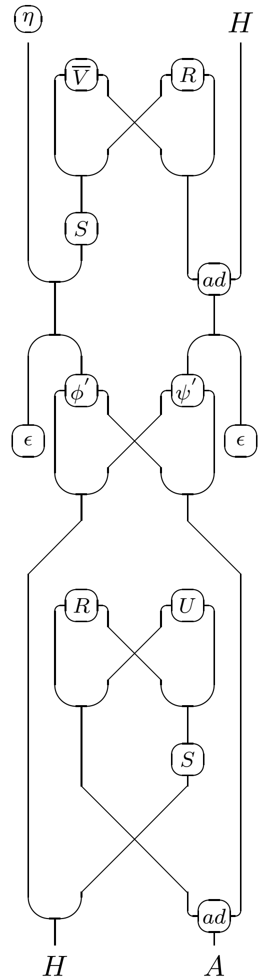
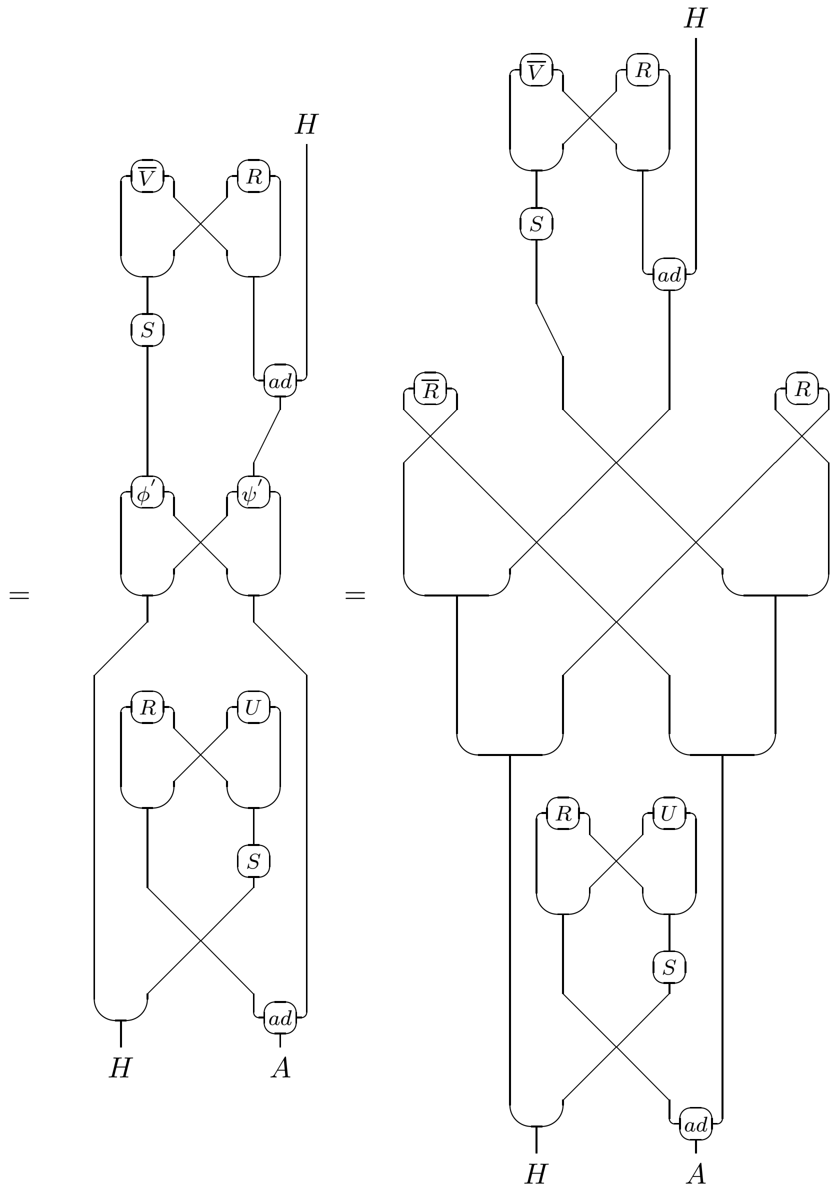
and ψ $\overset{\text{by proof of Th 1.2}}{=}$ $\zeta(\eta \otimes id) =$



by Th 1.4 (iii) $=$





Furthermore, $\psi = \psi'$ when H is commutative or $R = V$. In this case, $\underline{A}^{\phi'} \bowtie^{\psi'} \underline{H} = \underline{A^{\phi'} \bowtie^{\psi'} H} = \underline{A \bowtie^{R} H}$ as Hopf algebras living in braided tensor category $({}_D\mathcal{M}, C^{R_D})$.

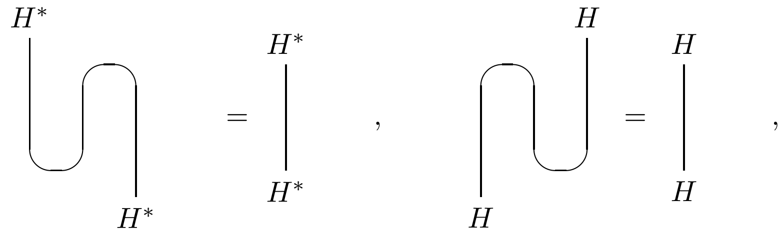
2 An example

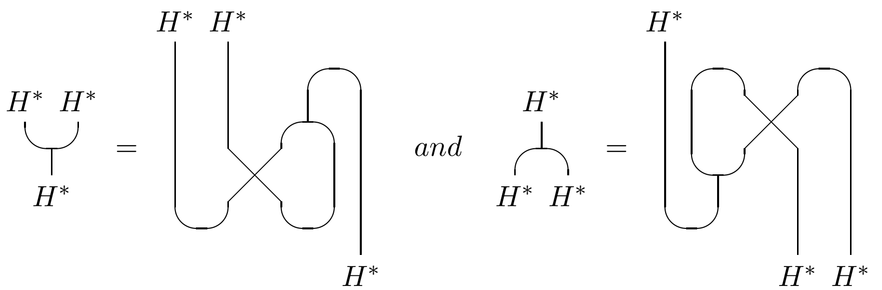
In this section, using preceding cofactorisation theorem, we give an example of double cross coproducts with both non-trivial left coaction and non-trivial right coaction in strictly braided tensor categories.

Let $H^* = Hom(H,k)$ be the dual of finite-dimensional Hopf algebra H. H^* can become a Hopf algebra under convolution ([10, Theorem 9.1.3]). That is, for any $f, g \in H^*, h, h' \in H$,

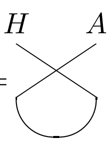
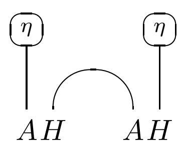
$$(f * g)(h) = \sum_{(h)} f(h_1)g(h_2), \Delta_{H^*}(f)(h \otimes h') = f(hh'), S_{H^*}(f)(h) = f(S(h)).$$

Assume $\{e_{x_i} \mid i = 1, 2, \cdots, n\}$ is the dual basis of $\{x_i \mid i = 1, 2, \cdots, n\}$. Define $d_H = \begin{cases} H^* \otimes H & \to k \\ f \otimes h & \to f(h) \end{cases}$ and $b_H = \begin{cases} k & \to H \otimes H^* \\ 1 & \to \sum_{i=1}^{n} x_i \otimes e_{x_i} \end{cases}$. d_H and b_H are called evaluation and coevoluation of H, respectively. It is clear that





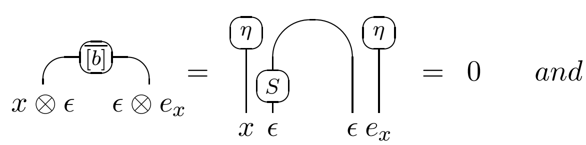
Note the multiplication and comultiplication of H^* exactly are anti-multiplication and anti-comultiplication in [8, Proposition 2.4].

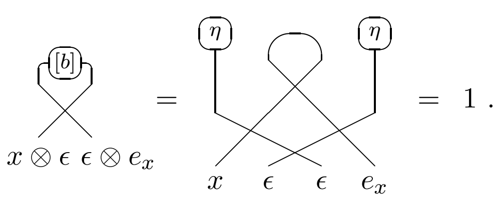
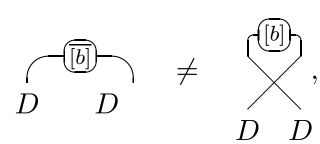
Let $A = H^{*cop}$, $\tau =$  *and* $[b] =$  .

Thus τ is a skew pairing and the Drinfeld Double $D(H) = A \bowtie_\tau H$ (see [4] [5]). Furthermore, $[b]$ is a quasitrianglar structure of $D(H)$ by[10, Theorem 10.3.6].

Lemma 2.1 *Let H be a finite-dimensional Hopf algebra with dim $H > 1$. Then Drinfeld double $(D(H), [b])$ is not triangular.*

Proof. Assume that $x_i's$ are a basis of H with $x_1 = 1_H$. Set $x =: x_2$ and see



 Thus 

which implies that $[b]$ is not triangular. □

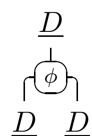
Let us recall Sweedler's four dimensional Hopf algebra H_4. That is, H_4 is a Hopf algebra generated g and x with relations

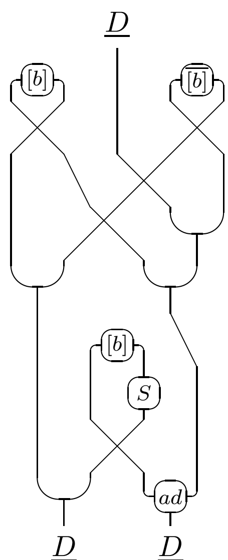
$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

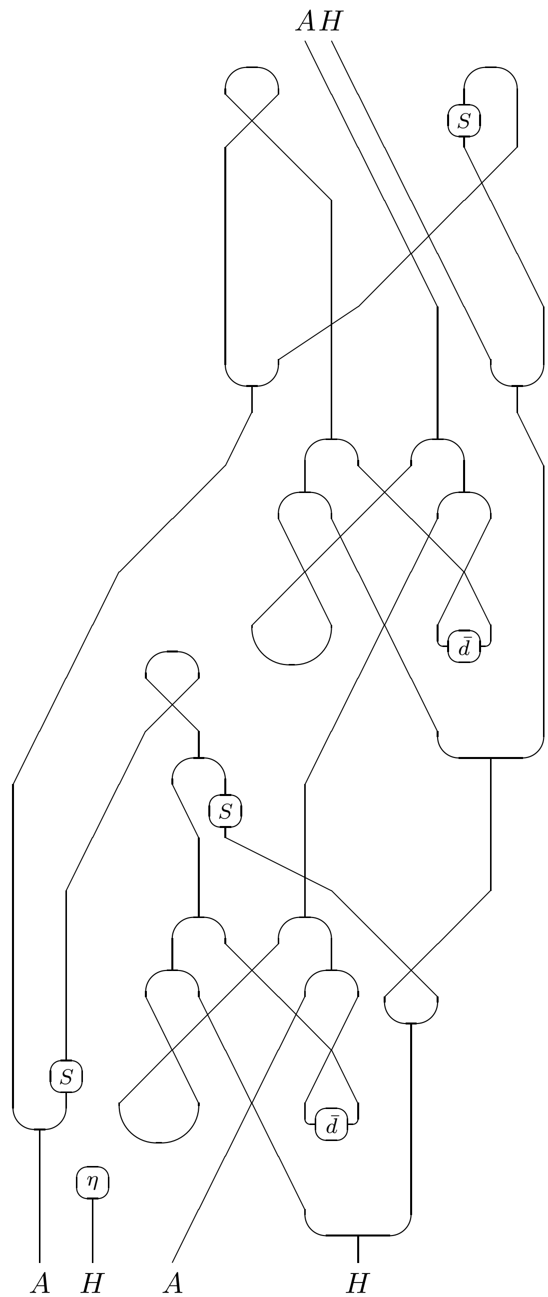
and $\Delta(x) = x \otimes 1 + g \otimes x$, $\Delta(g) = g \otimes g$, $\epsilon(x) = 0$, $\epsilon(g) = 1$, $S(x) = xg$, $S(g) = g$. Let $\{e_1, e_g, e_x, e_{gx}\}$ denote the dual basis of $\{1, g, x, gx\}$.

Example 2.2 *Let H be Sweedler's four dimensional Hopf algebra over field k with char $k \neq 2$. Let $D = D(H)$. Thus $B =: D \bowtie^{[b]} D$ is quasitriangular, but it is not triangular by Lemma 2.1. Considering [3, Theorem 2.5], B has a quasitriangular structure R_B, defined in preceding Theorem 1.4 with $U = V = 1 \otimes 1$, and R_B never is triangular . Thus $({}_B\mathcal{M}, C^{R_B})$ is a strictly braided tensor category by [10, Theorem 10.4.2 (3)]. It follows from Theorem 1.4 (ii) that $D \bowtie^{[b]} D \cong D^{\phi} \bowtie^{\psi} D$ for some ϕ and ψ. Furthermore, $D^{\phi} \bowtie^{\psi} D$ is a double cross coproduct. We shall show that both left coaction ϕ and right coaction ψ are non-trivial.*

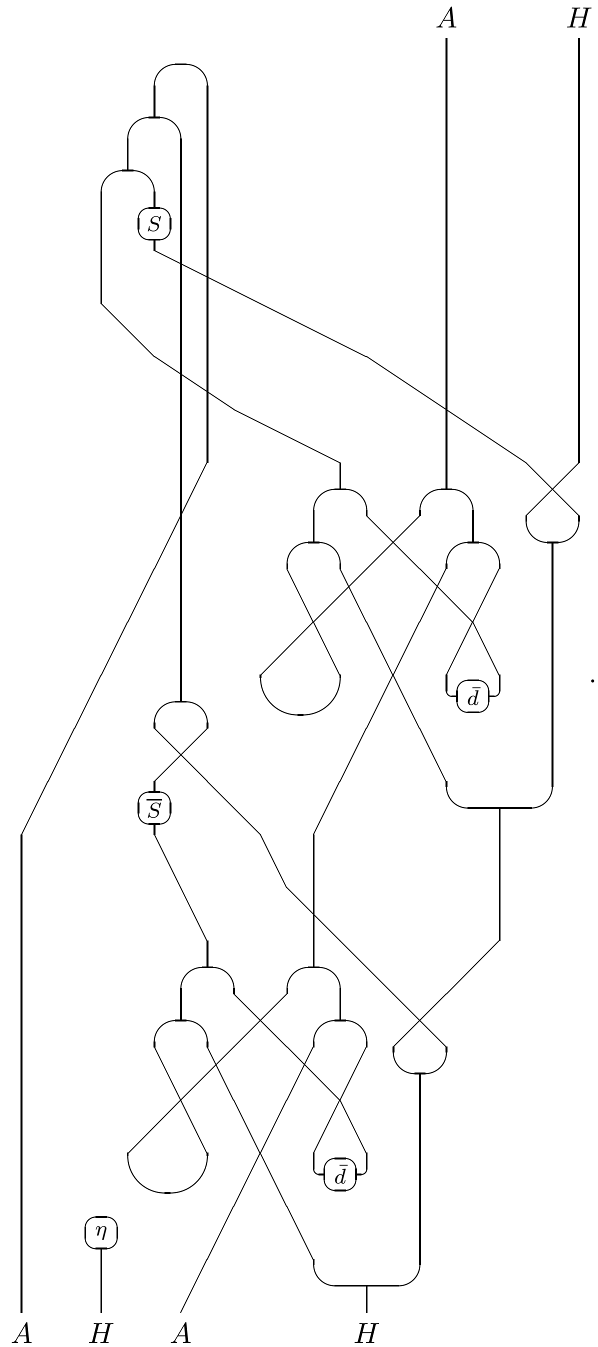
Proof.



$=$ 
D
[b]
[b]
[b]
S
ad
D
D

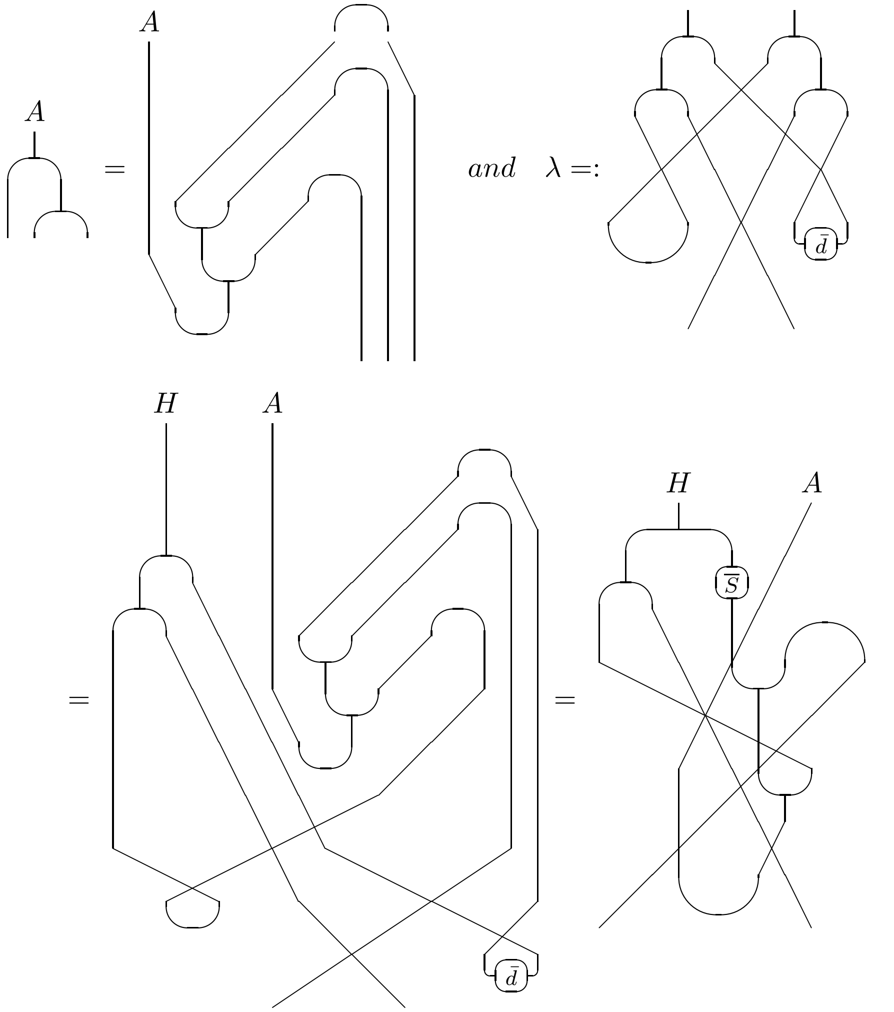
$=$ 
A H
S
d̄
S
S
d̄
η
A
H
A
H

=

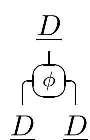

A
H
S
d̄
S̄
d̄
η
A
H
A
H

.

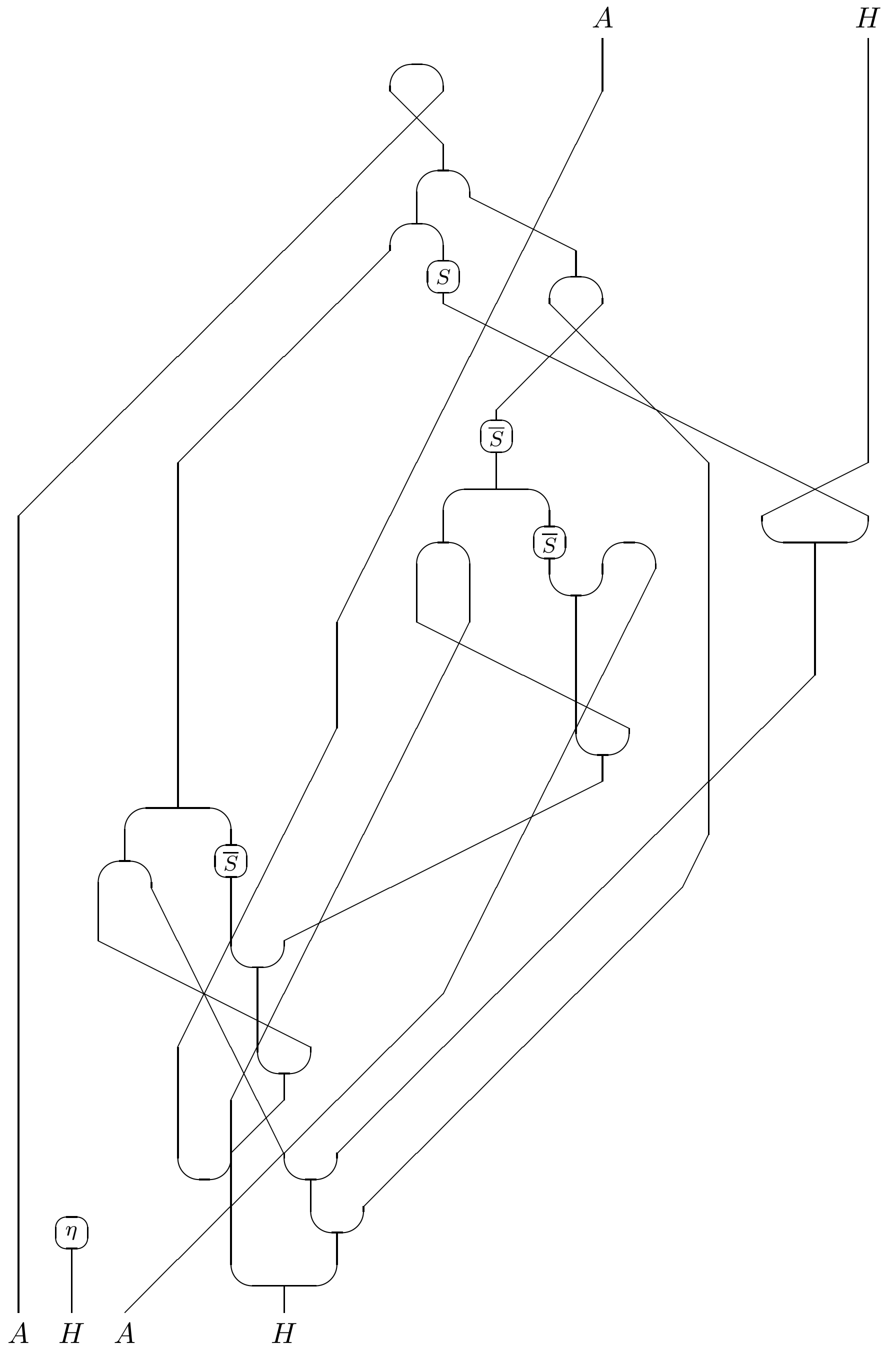
Compute:



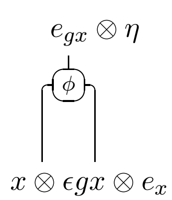
Also,



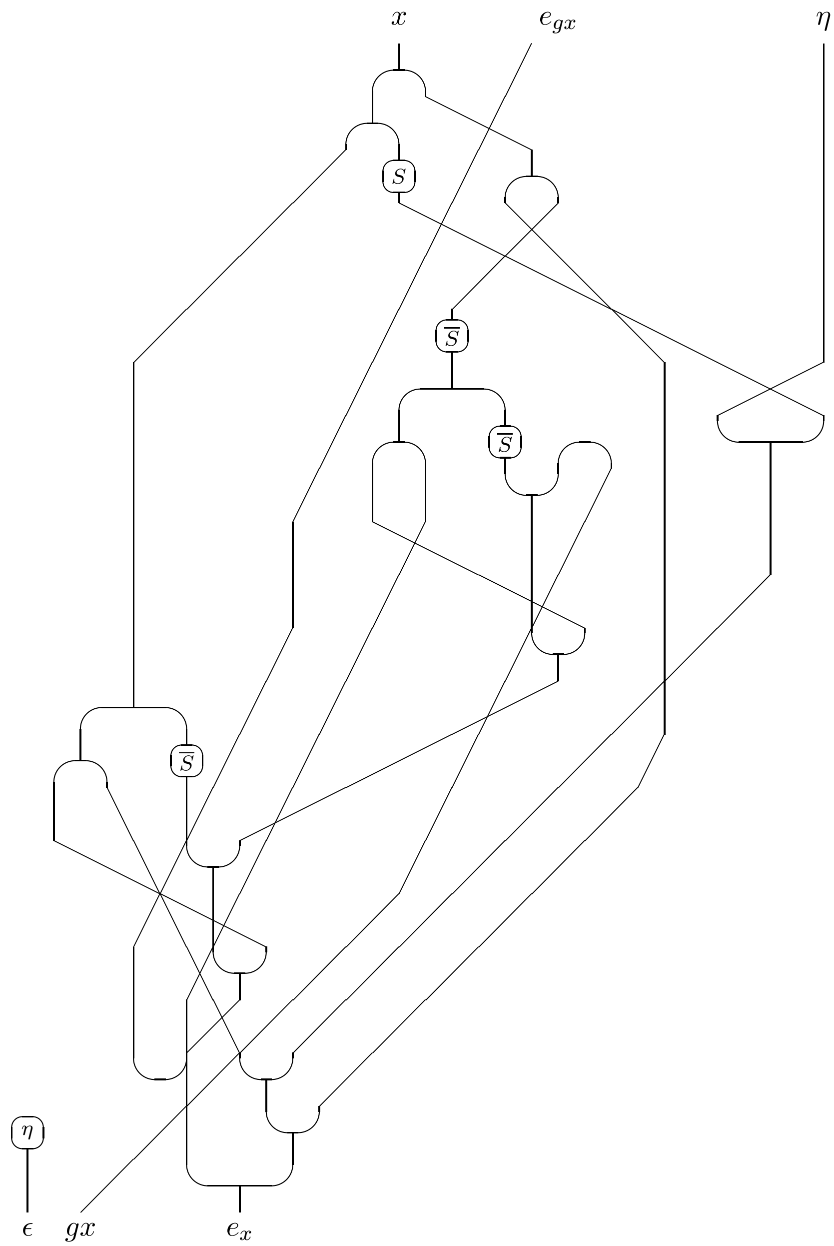
=


A
H
S
$\overline{S}$
$\overline{S}$
$\overline{S}$
η
A H A H

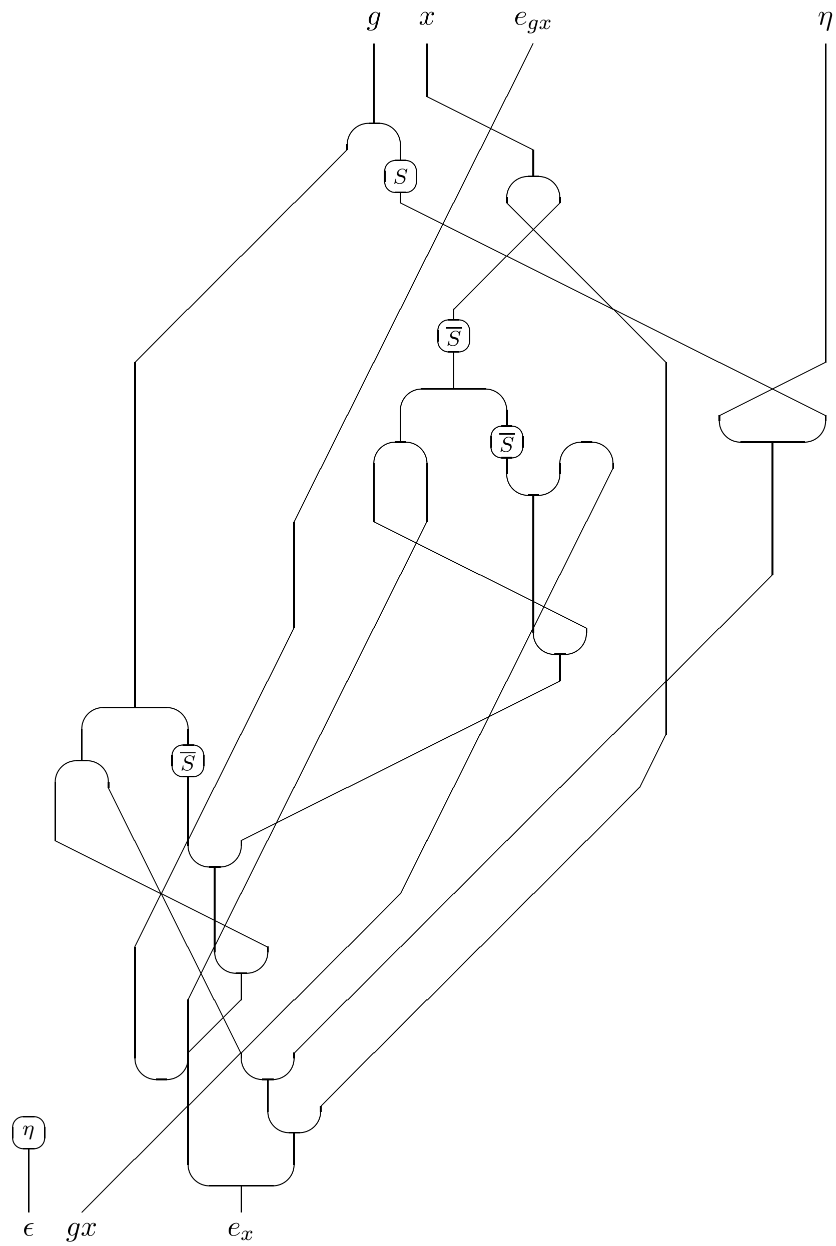
and


$e_{gx} \otimes \eta$
ϕ
$x \otimes \epsilon gx \otimes e_x$

=

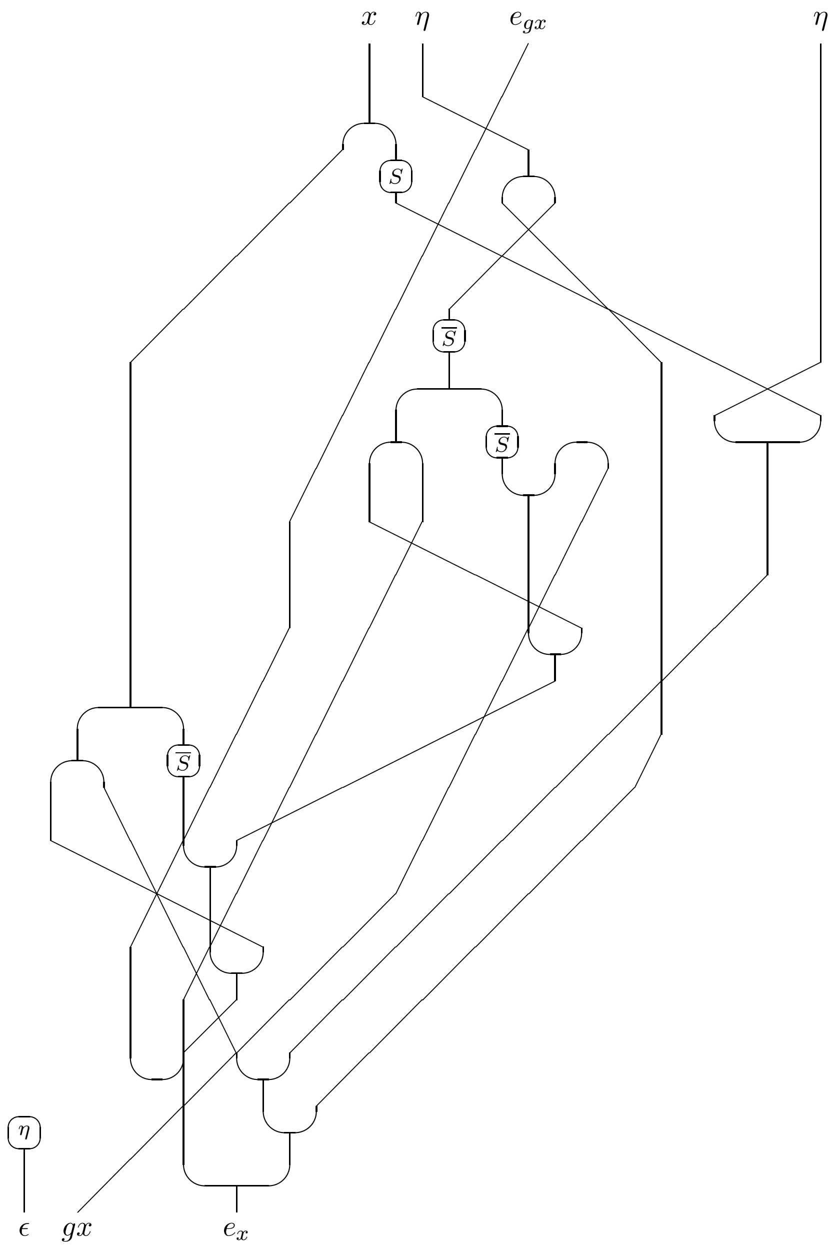

x
e_{gx}
η
S
$\overline{S}$
$\overline{S}$
$\overline{S}$
η
ϵ
gx
e_x

$=$



g
x
e_{gx}
η
S
$\overline{S}$
$\overline{S}$
$\overline{S}$
η
ϵ
gx
e_x

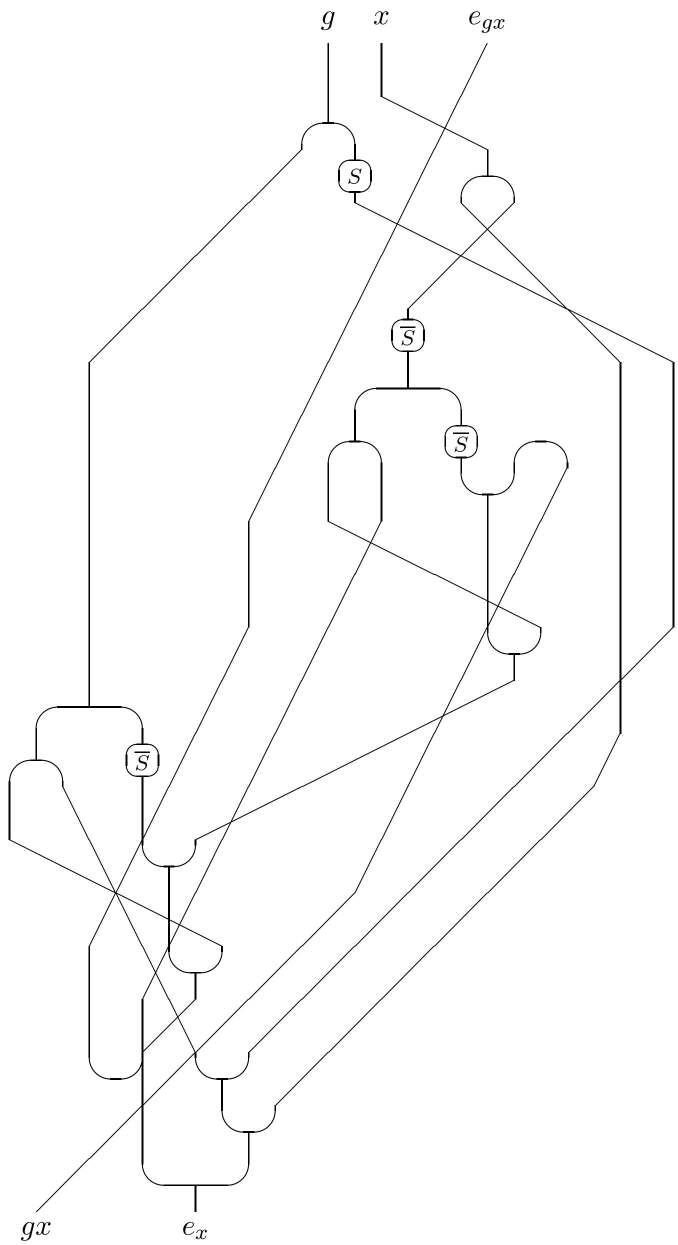
+



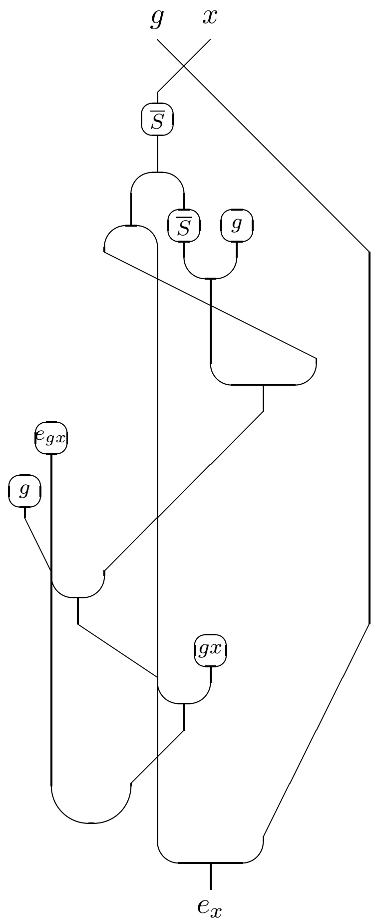
x
η
e_{gx}
η
S
$\overline{S}$
$\overline{S}$
$\overline{S}$
η
ϵ
gx
e_x

Let u and v denote the first term and the second term, respectively.

$u =$

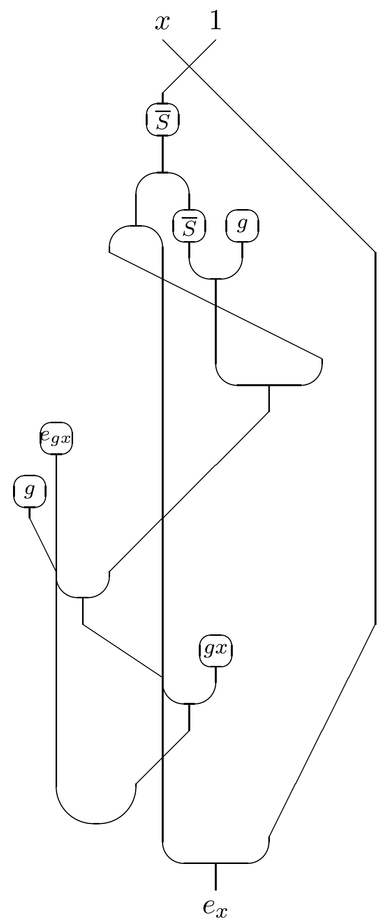


$=$

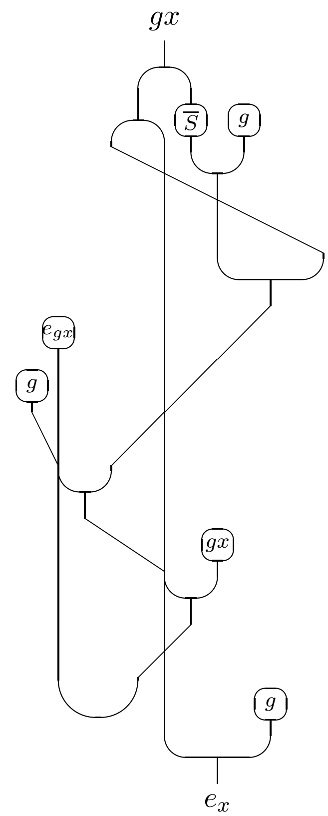


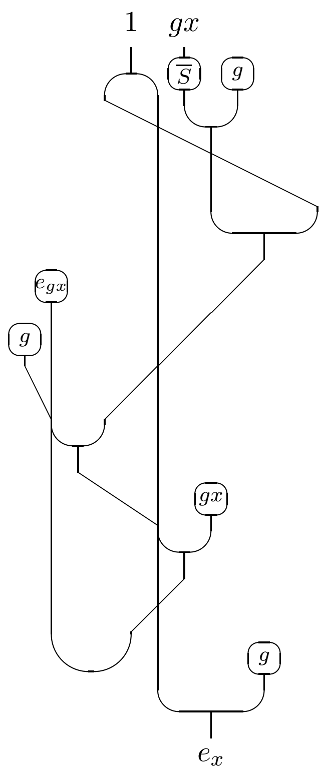
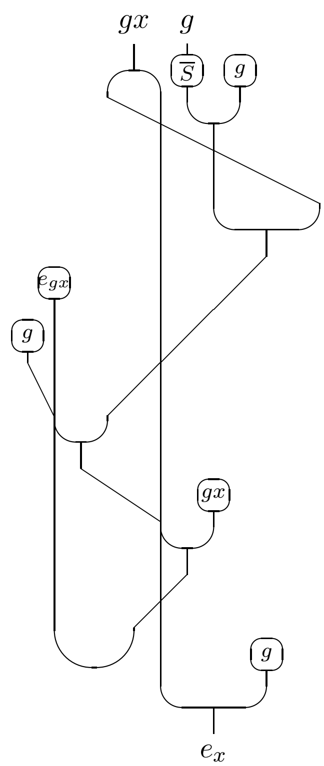
g
x
S̄
S̄
g
e_gx
g
gx
e_x

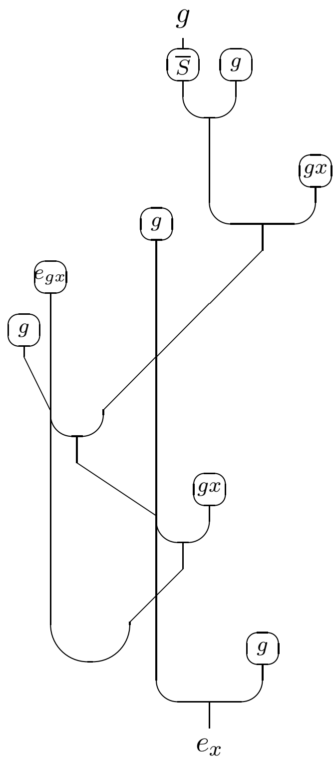
$+$



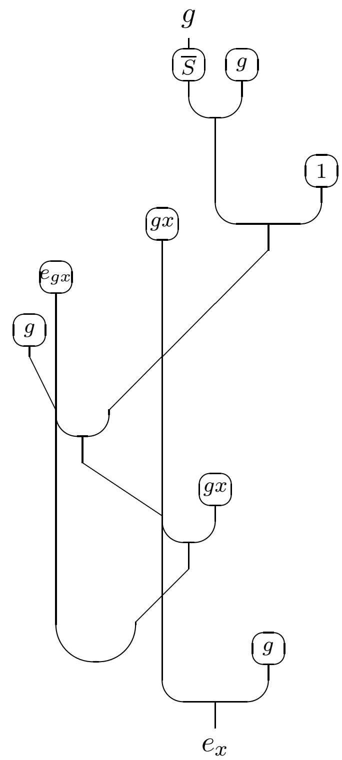
x
1
S̄
S̄
g
e_gx
g
gx
e_x

$=$  $+1$

$=$  $+$  $+1$



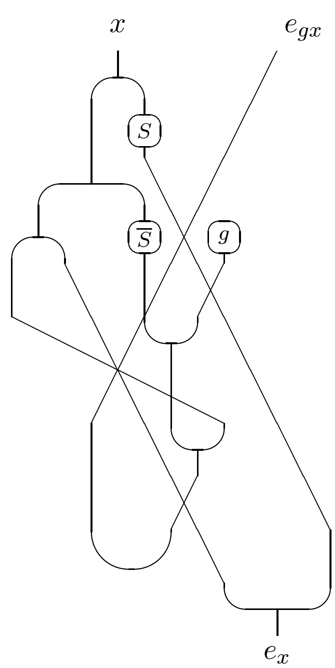
$+$



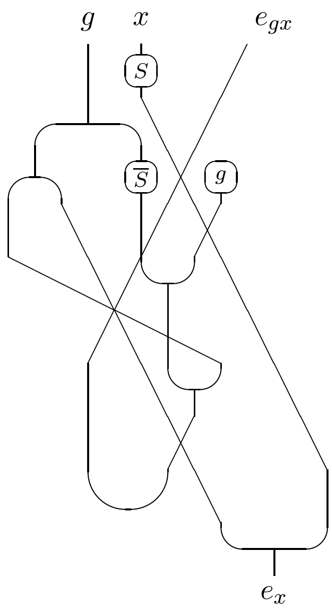
$+1 \;\; = 1$

and

$v \;=$

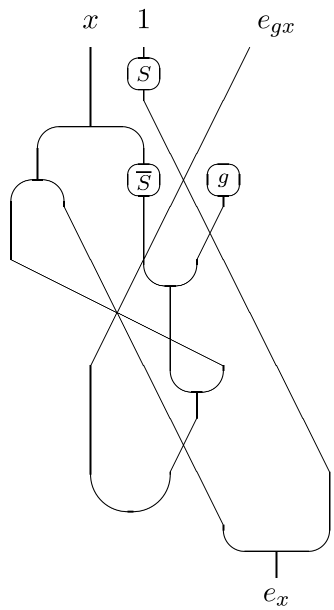


=



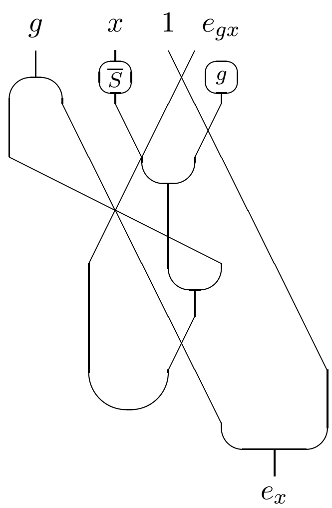
g
x
e_{gx}
S
$\overline{S}$
g
e_x

+



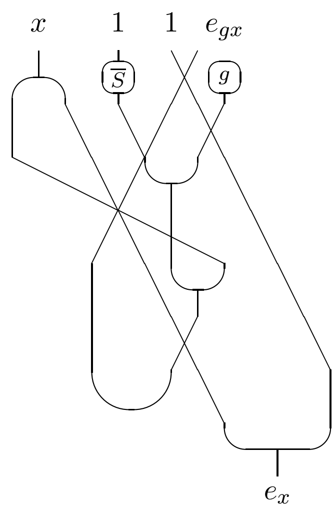
x
1
e_{gx}
S
$\overline{S}$
g
e_x

$= 1 +$



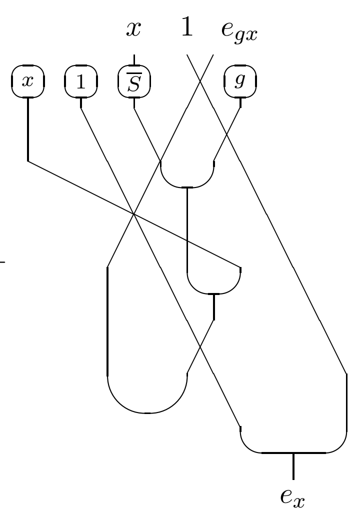
g
x
1
e_{gx}
$\overline{S}$
g
e_x

+



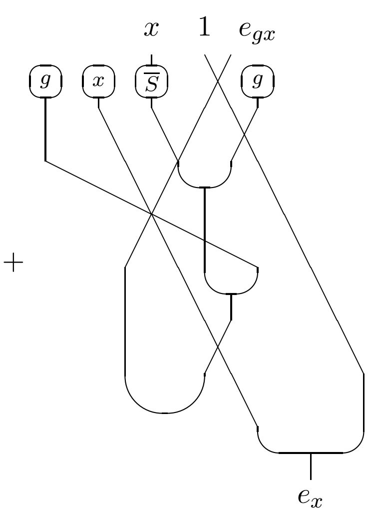
x
1
1
e_{gx}
$\overline{S}$
g
e_x

$= 1 +$



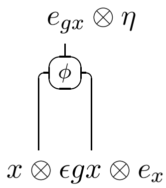
x
1
e_{gx}
x
1
$\overline{S}$
g
e_x

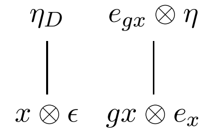
+



x
1
e_{gx}
g
x
$\overline{S}$
g
e_x

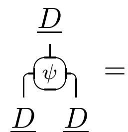
$= \; 1.$

Thus 

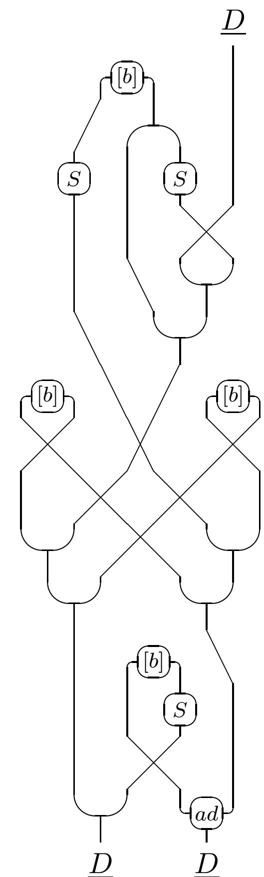
$= 2,$ *but* 

$= 0.$

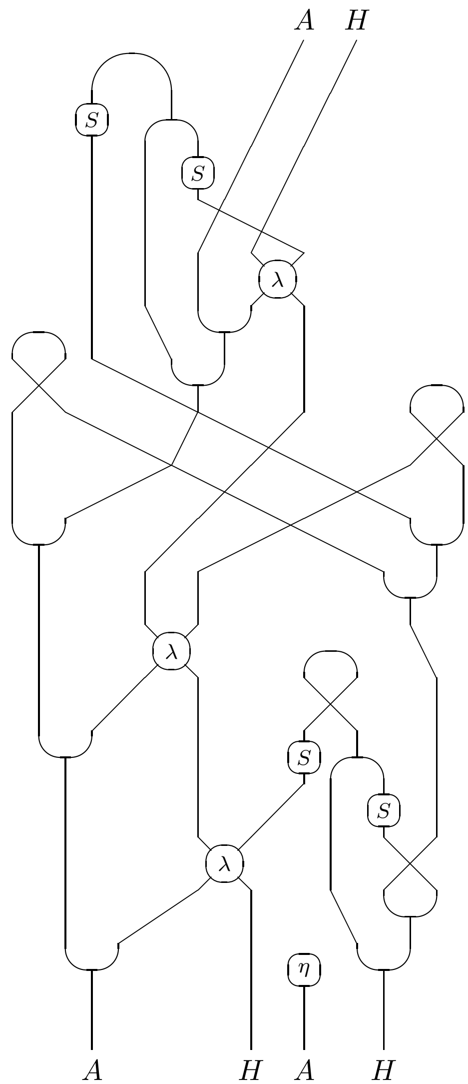
Consequently, ϕ is not trivial.



$=$

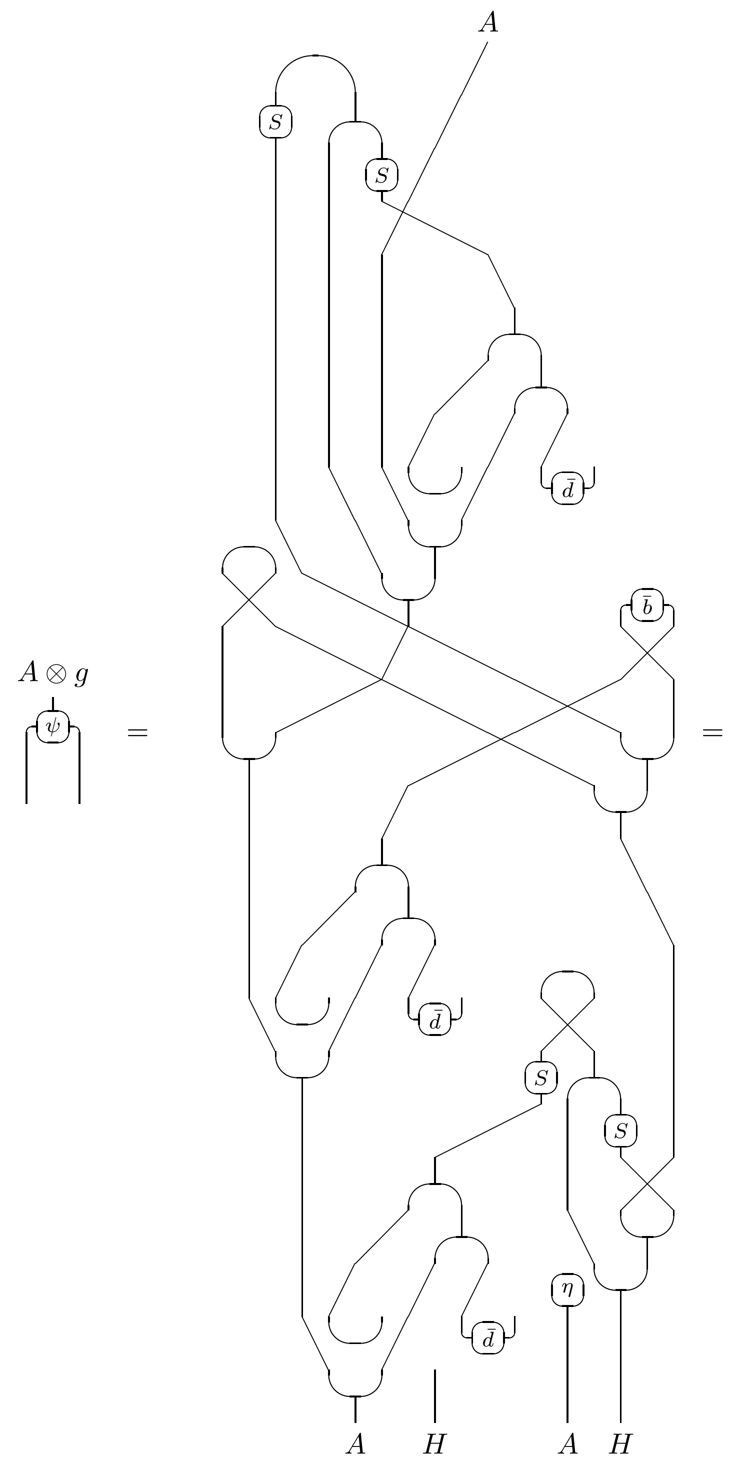


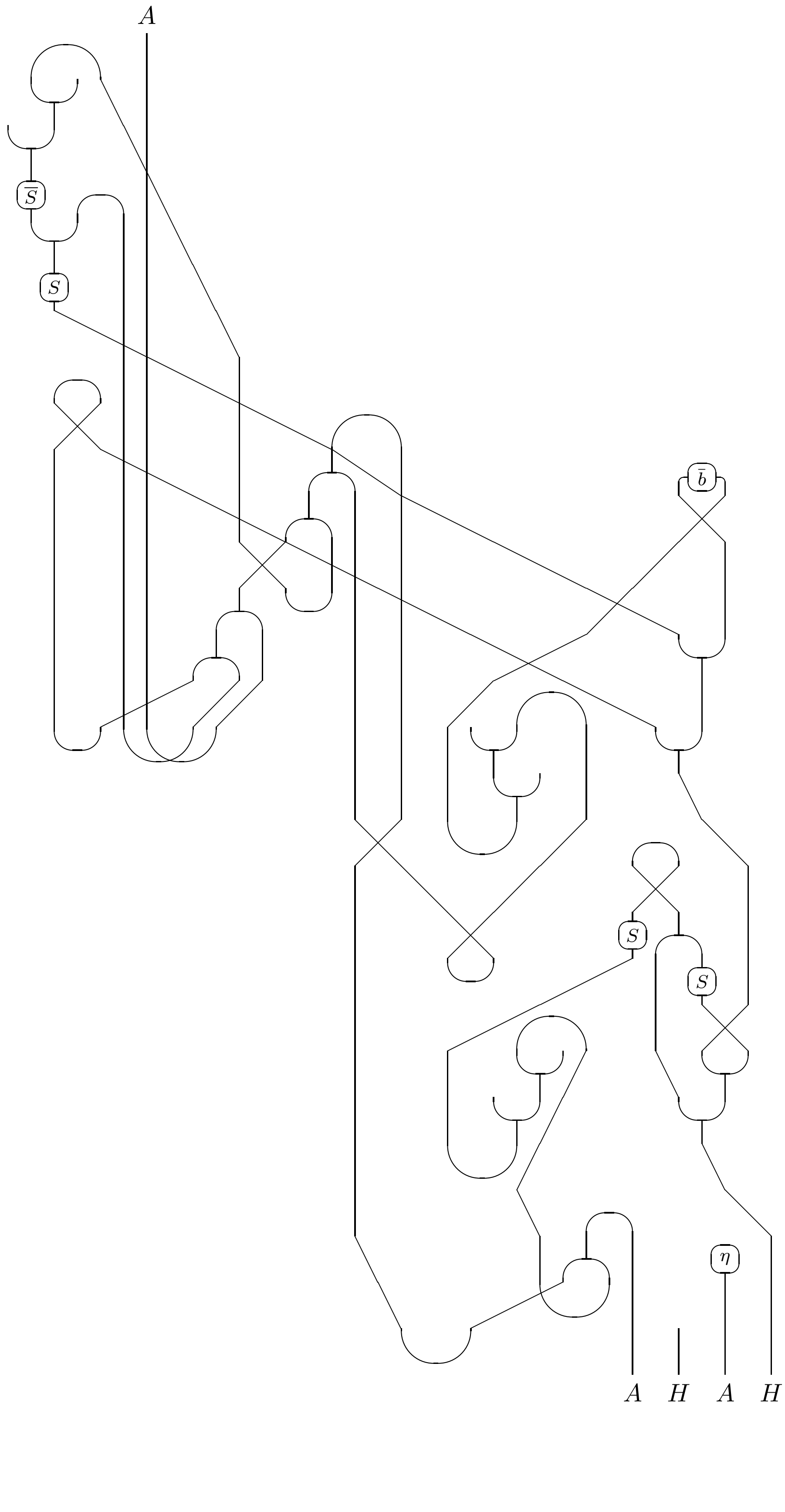
$=$



.

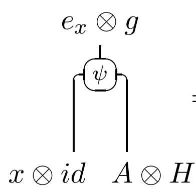
For convenience, we omit g in the following diagrams:



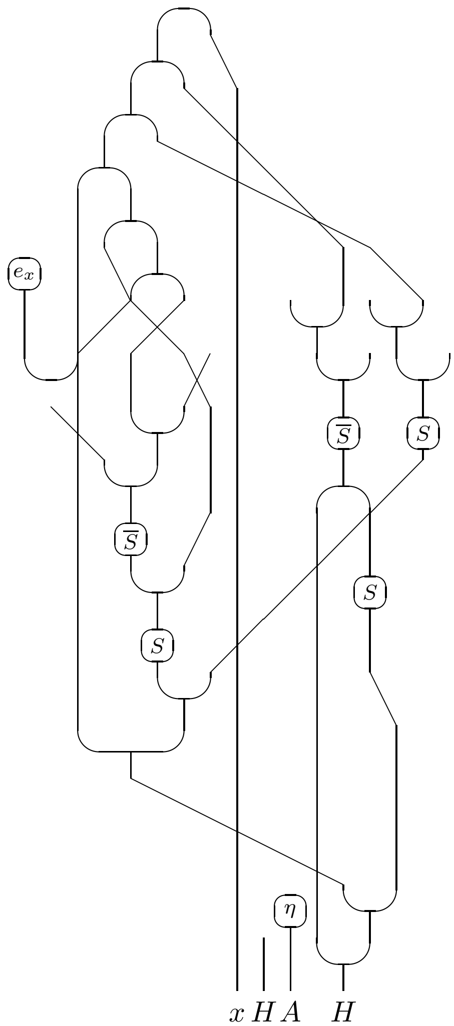


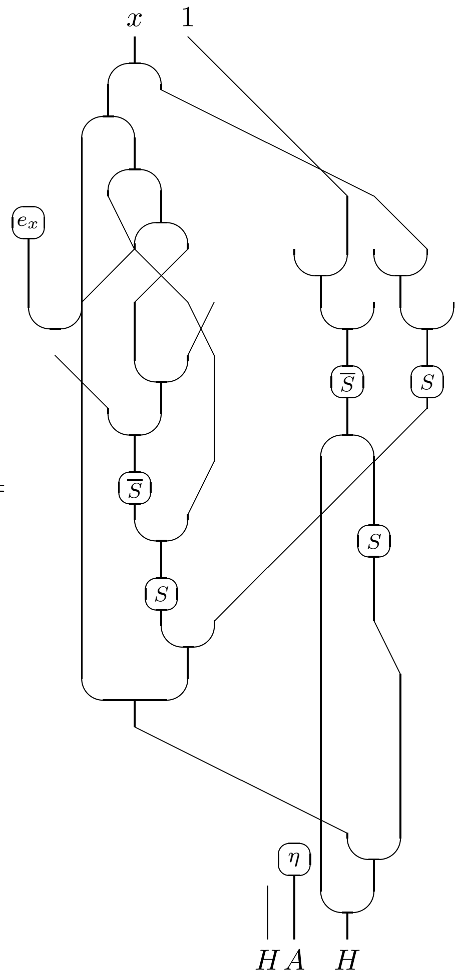
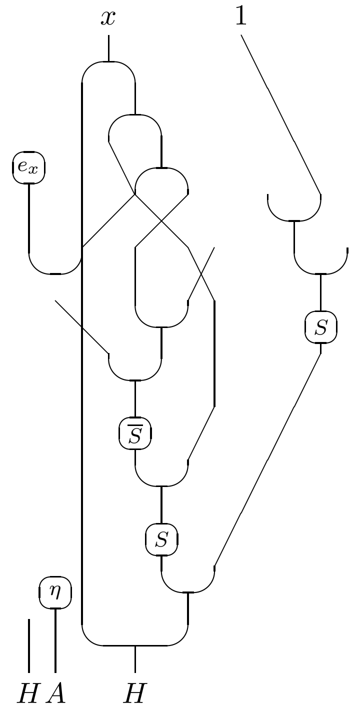
A
$\overline{S}$
S
$\bar{b}$
S
S
η
A
H
A
H

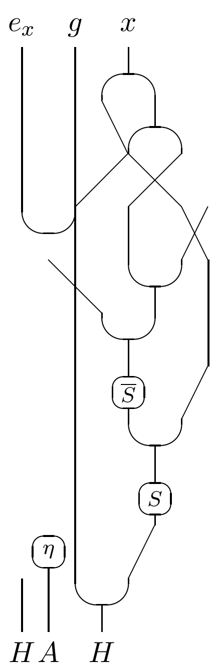
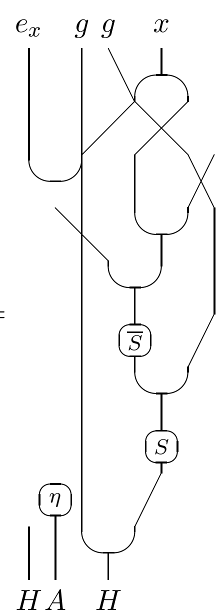
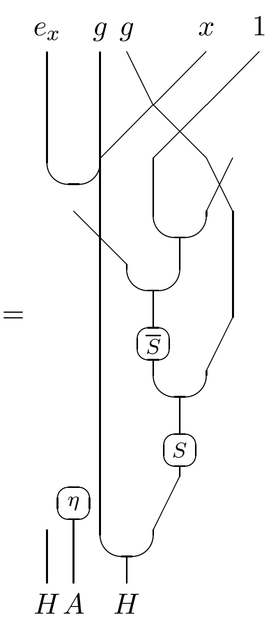
and

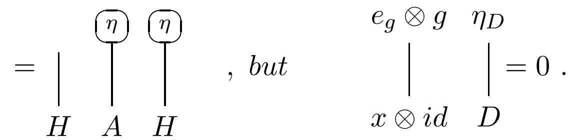

$e_x \otimes g$
ψ
$x \otimes id$
$A \otimes H$

=


e_x
$\overline{S}$
S
$\overline{S}$
S
S
η
x H A H

= 
x
1
e_x
S̄
S
η
H A H
= 
x
1
e_x
S̄
S
η
H A H

= 
e_x g x
S̄
S
η
H A H
= 
e_x g g x
S̄
S
η
H A H
= 
e_x g g x 1
S̄
S
η
H A H



Thus ψ is not trivial. $\square$

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